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Continued-fraction solutions to the Riccati equation and integrable lattice systems

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Abstract. Continued-fraction solutions to the matrix Riccati equation are discussed which are constructed by using the concept of *form invariance*. It is demonstrated that this technique is related to the AKNS method of deriving integrable nonlinear lattice systems. This gives an explanation why continued-fraction solutions related to the Toda lattice were obtained in a previous work.

Continued fractions corresponding to Kac-Van Moerbeke, discrete nonlinear Schrödinger and discrete modified KdV lattice equations are constructed. A method for linearising the Kac-Van Moerbeke lattice equations is rederived and particular solutions are generated. Our approach demonstrates the crucial role played by the boundary condition at the finite end of the lattice for the existence of this method. These results are extended to the other two lattice systems above in the semi-infinite case and corresponding particular solutions generated in terms of Bessel functions.

1. Introduction

In a previous study (Common and Roberts 1986) continued-fraction solutions to the matrix Riccati equation (MRE)

$$\dot{Z}_0(t) = E_0(t) + G_0(t)Z_0(t) + Z_0(t)F_0(t) + Z_0(t)H_0(t)Z_0(t) \tag{1.1}$$

were considered, where for simplicity we take Z_0, E_0, etc, to be $(n \times n)$ matrices. They were generated by making the sequence of substitutions,

$$Z_k(t) = U_0 + [N_{k+1}(t) - Z_{k+1}(t)]^{-1} M_{k+1}(t) \qquad k = 0, 1, 2, \ldots \tag{1.2}$$

where $N_k(t)$, $M_k(t)$ are $(n \times n)$ matrix functions of t and U_0 is an $(n \times n)$ constant matrix. The elements N_k, M_k of the resulting continued-fraction for Z_0 were determined by requiring that all the Z_k satisfied a MRE:

$$\dot{Z}_k(t) = E_k(t) + G_k(t)Z_k(t) + Z_k(t)F_k(t) + Z_k(t)H_k(t)Z_k(t) \qquad k = 0, 1, 2, \ldots \tag{1.3}$$

with the same *standard form* $G_k \equiv 0$, $H_k \equiv 1$. This requirement is satisfied when U_0 is proportional to the unit matrix if

$$\dot{N}_k(t) = M_{k+1}(t) - M_k(t) \tag{1.4a}$$

$$\dot{M}_{k+1}(t)M_{k+1}^{-1}(t) = N_{k+1}(t) - M_{k+1}(t)N_k(t)M_{k+1}^{-1} \qquad k = 1, 2, \ldots. \tag{1.4b}$$

These are equations for a Toda lattice (Toda 1976) and its matrix generalisation (Bruschi *et al* 1980).

In section 2 we will demonstrate that this unexpected result arises through a connection with the AKNS method (Ablowitz *et al* 1973, 1974) for constructing nonlinear evolution equations solvable by the inverse scattering method. In the following sections, various examples of these equations will be considered by taking different *standard forms* of the MRE in (1.1) and other forms of linear-fractional transformation corresponding to (1.2).

In section 3 the scalar lattice introduced by Kac and Van Moerbeke (1975) is discussed and we suggest how it may be generalised to the matrix case. It will be shown that the relation with the continued-fraction solution of the MRE (1.1) provides a method for linearising these equations. This method differs from the usual inverse scattering technique and is, for the scalar case, equivalent to that introduced recently by Yamazaki (1987).

Our approach illustrates the important role that a boundary condition of the half-infinite lattice plays in the existence of the above linearisation method. We use it to construct special rational solutions of the Kac–Van Moerbeke (KVM) lattice equations. They are related in a standard way to solutions of an half-infinite Toda lattice.

Using the connection of our method with the AKNS approach, we extend our results to the discrete nonlinear Schrödinger equation (DNLS) in section 4 and a discrete modified KdV equation (DMKdV) in section 5, both on a half-infinite interval. Here again a linearisation of these equations is obtained, which is different from the usual one, if a required boundary condition at the finite end is satisfied. Special solutions are constructed which are different from those given by the standard inverse scattering method.

2. Continued-fractions from the linearised Riccati equation

The MRE (1.3) is equivalent to the pair of linear equations

$$\begin{aligned} \dot{X}_k &= G_k X_k + E_k Y_k \\ \dot{Y}_k &= -H_k X_k - F_k Y_k \qquad k=0,1,2,\ldots \end{aligned} \tag{2.1}$$

when $Z_k = X_k Y_k^{-1}$, whilst the linear fractional transformation (1.2) with $U_0 = \lambda 1$ is equivalent to

$$\left.\begin{aligned} X_{k+1} &= N_{k+1}X_k - [M_{k+1} + \lambda N_{k+1}]Y_k \\ Y_{k+1} &= X_k - \lambda Y_k \end{aligned}\right\} \qquad k=0,1,2,\ldots. \tag{2.2}$$

Differentiating (2.2) and using (2.1), we find X_{k+1}, Y_{k+1} satisfy (2.1) with $k \to k+1$ when

$$\begin{aligned} G_{k+1} = {} & \dot{M}_{k+1}M_{k+1}^{-1} - \lambda N_{k+1}G_k M_{k+1}^{-1} - \lambda M_{k+1}H_k M_{k+1}^{-1} \\ & - \lambda^2 N_{k+1}H_k M_{k+1}^{-1} - N_{k+1}E_k M_{k+1}^{-1} \\ & - M_{k+1}F_k M_{k+1}^{-1} - \lambda N_{k+1}F_k M_{k+1}^{-1} \end{aligned} \tag{2.3a}$$

$$\begin{aligned} E_{k+1} = {} & \dot{N}_{k+1} + N_{k+1}G_k + M_{k+1}H_k + \lambda N_{k+1}H_k \\ & + \lambda N_{k+1}G_k M_{k+1}^{-1}N_{k+1} - \dot{M}_{k+1}M_{k+1}^{-1}N_{k+1} + \lambda M_{k+1}H_k M_{k+1}^{-1}N_{k+1} \\ & + \lambda^2 N_{k+1}H_k M_{k+1}^{-1}N_{k+1} + N_{k+1}E_k M_{k+1}^{-1}N_{k+1} + M_{k+1}F_k M_{k+1}^{-1}N_{k+1} \\ & + \lambda N_{k+1}F_k M_{k+1}^{-1}N_{k+1} \end{aligned} \tag{2.3b}$$

$$H_{k+1}=E_kM_{k+1}^{-1}+\lambda G_kM_{k+1}^{-1}+\lambda^2H_kM_{k+1}^{-1}+\lambda F_kM_{k+1}^{-1} \tag{2.3c}$$

$$F_{k+1}=-G_k-E_kM_{k+1}^{-1}N_{k+1}-\lambda H_k-\lambda G_kM_{k+1}^{-1}N_{k+1}$$
$$-\lambda^2H_kM_{k+1}^{-1}N_{k+1}-\lambda F_kM_{k+1}^{-1}N_{k+1}. \tag{2.3d}$$

Restricting the MRE (1.3) to a given *standard form* for $k=0,1,2,\ldots$ leads to relations between the M_k and N_k which take the form of nonlinear equations. For instance, if we require $G_k\equiv 0$, $H_k\equiv 1$ for all $k=0,1,\ldots,$ we recover a previous result (Common and Roberts 1986) that M_k, N_k must be solutions of the Toda lattice equations. On the other hand, if we require $H_k\equiv 1$, $U_0=0$, $G_0=0$, $N_k=\mu 1$ where μ is a scalar constant then the M_k must satisfy

$$\mu\dot{M}_{k+1}(t)=M_{k+1}(t)M_k(t)-M_{k+2}(t)M_{k+1}(t)+[M_1(t),M_{k+1}(t)]$$
$$k=1,2,\ldots. \tag{2.4}$$

We will see in section 3 that in the scalar case this set of equations are essentially those for the KVM lattice.

The AKNS method may be used to generate the above evolution equations and many others in the scalar case. A good survey of the method has been given by Ablowitz (1978). It involves a pair of coupled evolution equations:

$$\dot{\nu}_{1k}(t)=A_k\nu_{1k}(t)+B_k\nu_{2k}(t) \tag{2.5a}$$

$$\dot{\nu}_{2k}(t)=C_k\nu_{1k}(t)+D_k\nu_{2k}(t) \tag{2.5b}$$

and an eigenvalue problem:

$$\nu_{1k+1}(t)=\lambda\nu_{1k}+Q_k\nu_{2k}+S_k\nu_{2k+1} \tag{2.6a}$$

$$\nu_{2k+1}(t)=(1/\lambda)\nu_{2k}+R_k\nu_{1k}+T_k\nu_{1k+1} \tag{2.6b}$$

where the eigenvalue λ is independent of t, k.

By choosing suitable forms for the Q_k, R_k, S_k and T_k as functions of t and taking corresponding parametrisations of A_k, B_k, C_k and D_k as functions of λ, t, various nonlinear lattice equations can be obtained as consistency conditions for (2.5) and (2.6). For example, setting

$$R_k\equiv 0 \qquad T_k\equiv 1 \qquad Q_k\equiv-\beta_k(t) \qquad S_k\equiv[1-\alpha_k(t)] \tag{2.7a}$$

$$A_k\equiv\lambda-\beta_k \qquad B_k\equiv\beta_k+(1-\alpha_{k-1})/\lambda \qquad C_k\equiv\lambda \qquad D_k\equiv 1/\lambda \tag{2.7b}$$

then (2.5) and (2.6) are consistent if and only if

$$\dot{\beta}_k(t)=\alpha_{k-1}(t)-\alpha_k(t) \tag{2.8a}$$

$$\dot{\alpha}_k(t)\alpha_k^{-1}(t)=\beta_k(t)-\beta_{k+1}(t). \tag{2.8b}$$

Making the substitutions $k\to-(+k+1)$, $\alpha_{-k}\to M_k$, $\beta_{-k}\to N_k$, we see immediately that these consistency conditions are equivalent to those for the Toda lattice (1.4*a*, *b*).

It is now easy to see how our approach for generating continued-fraction solutions to the MRE corresponds to the AKNS method. First of all the evolution equations (2.5) and (2.1) are essentially of the same form. Also the eigenvalue equations (2.6*a*, *b*) can be solved for ν_{1k+1}, ν_{2k+1} to give linear equations which are of the same form as (2.2). Our equations for the elements of the continued fraction for Z_0 are obtained by requiring consistency of (2.1) with (2.2), so the connection with the AKNS method is complete.

3. The Kac–Van Moerbeke lattice

We have seen in the previous section that, taking the MRE (1.3) with $H_k \equiv 1$ and additionally $G_0 \equiv 0$, $N_k = \mu 1$, then the elements of the corresponding matrix continued fraction:

$$Z_0(t) = [\mu 1 - [\mu 1 - [\mu 1 - \ldots]^{-1} M_3]^{-1} M_2]^{-1} M_1 \tag{3.1}$$

is a solution of (1.1) when the M_k satisfy the relations (2.4). In the scalar case, setting $M_k(-t) = \mu y_k(t)$, they become

$$\dot{y}_k(t) = y_k(t)[y_{k+1}(t) - y_{k-1}(t)] \qquad k = 2, 3, 4, \ldots. \tag{3.2}$$

These relations, with the added boundary conditions

$$\dot{y}_1(t) = y_1(t) y_2(t) \qquad y_k \to 0 \text{ as } k \to \infty \tag{3.3}$$

are the set of equations for the nonlinear lattice investigated by Kac and Van Moerbeke (1975). Making a similar substitution in the matrix case by setting $M_k(-t) = \mu Y_k(t)$, in (2.4), they become

$$\dot{Y}_k(t) = Y_{k+1}(t) Y_k(t) - Y_k(t) Y_{k-1}(t) + [Y_k(t), Y_1(t)] \qquad k = 2, 3, \ldots. \tag{3.4}$$

We suggest that these equations with the added boundary conditions

$$\dot{Y}_1(t) = Y_2(t) Y_1(t) \qquad Y_k \to 0 \text{ as } k \to \infty \tag{3.5}$$

give the matrix generalisation of the Kac–Van Moerbeke lattice.

Recently Yamazaki (1987) proposed a method for linearising the nonlinear equations (3.2) and (3.3) which does not require the boundary condition $y_k \to 0$ as $k \to \infty$ used in the standard *inverse scattering technique.* We will now use our formulation to rederive Yamazaki's result and at the same time extend it to the matrix case.

For the standard form of MRE considered in this section, it is straightforward to obtain E_0 and F_0 from (2.3) and hence show that in this case (1.1) has the form,

$$\begin{aligned}\dot{Z}_0(t) = M_1(t) + Z_0(t)[(1/\mu) M_1^{-1}(t) M_2(t) M_1(t) \\ - (1/\mu) M_1(t) + M_1^{-1}(t) \dot{M}_1(t) - \mu 1] + Z_0(t)^2.\end{aligned} \tag{3.6}$$

Consider now the matrix continued fraction

$$\begin{aligned}W(t) &= \sqrt{\mu}[\mu 1 - Z_0(-t)]^{-1} \\ &= [\lambda 1 - [\lambda 1 - [\lambda 1 \ldots]^{-1} Y_2(t)]^{-1} Y_1(t)]^{-1}\end{aligned} \tag{3.7}$$

where $\lambda = \sqrt{\mu}$, the Y_k satisfy (3.4) and (3.5) and a standard equivalence relation for continued fractions has been used with (3.1). We have the following result.

Theorem 3.1. The matrix coefficients in the power series expansion

$$W(t) = \sum_{k=0}^{\infty} \omega_k(t)/\lambda^{2k+1} \qquad \omega_0(t) = 1 \tag{3.8}$$

obey the *linear* relations:

$$\dot{\omega}_k(t) = \omega_{k+1}(t) - Y_1(t) \omega_k(t) \qquad k = 0, 1, 2, \ldots. \tag{3.9}$$

Proof. From (3.7)

$$Z_0(-t)=\lambda^2-\lambda/W(t) \tag{3.10}$$

and substituting into (3.6)

$$\begin{aligned}\dot{W}(t)=-\lambda 1+[Y_1^{-1}Y_2Y_1-Y_1-Y_1^{-1}\dot{Y}_1+\lambda^2 1]W(t)\\ -\lambda W(t)[Y_1^{-1}Y_2Y_1-Y_1^{-1}\dot{Y}_1]W(t).\end{aligned} \tag{3.11}$$

We now see that the *boundary condition* (3.5) ensures that the term quadratic in W vanishes. Substituting the power series (3.8) in (3.11) then gives the *linear* relations (3.9) on equating powers of $1/\lambda^2$.

Since the elements $Y_k(t)$ of the continued fraction expansion (3.7) for $W(t)$ may be expressed in terms of the coefficients $\omega_k(t)$ of its power series expansion, as we will discuss below, the *linear* differential equations (3.9) give a *linearisation* of the nonlinear equations (3.4) and (3.5). For the scalar case this is just the result of Yamazaki (1987).

Our method extends his result to the matrix case and also illustrates the crucial role of the above *boundary* condition.

For the scalar case there are well known expressions for $Y_k(t)$ in terms of determinants with elements depending on $\omega_l(t)$, $0\leqslant l\leqslant 2k-1$ (Jones and Thron 1980). They are

$$Y_k=P_{k-2}P_{k+1}/P_{k-1}P_k \qquad k=0,1,2,\ldots \tag{3.12}$$

with

$$P_0=P_{-1}=1$$

and

$$P_{2k-1}=H_k^{(0)} \qquad P_{2k}=H_k^{(1)} \qquad k=1,2,3,\ldots \tag{3.13}$$

where

$$H_m^{(n)}=\begin{vmatrix}\omega_n & \cdots & \omega_{n+m-1}\\ \vdots & \cdots & \vdots\\ \omega_{n+m-1} & \cdots & \omega_{n+2m-2}\end{vmatrix} \tag{3.14}$$

are Hankel determinants. Generalisations to the matrix case have been discussed by Wynn (1963).

Yamazaki has demonstrated that the *linear* differential equations (3.9) may be further simplified by making the substitution

$$\omega_k(t)=\delta_1^{-1}(t)\delta_{k+1} \qquad k=1,2,3,\ldots. \tag{3.15}$$

Then (3.15) gives a solution of (3.9) when

$$\dot{\delta}_k(t)=\delta_{k+1}(t) \qquad k=1,2,3,\ldots. \tag{3.16}$$

This may be proved trivially in both scalar and matrix cases using the identity $Y_1(t)=\omega_1(t)$.

That we have linearised the KVM lattice equations is illustrated by the following result for the scalar case.

Theorem 3.2. Let $\delta_k^{(i)}$; $k=1,2,3,\ldots$, $i=1,2$, be the two sets of solutions of (3.16) corresponding to boundary values $y_1^{(i)}(t)$, $i=1,2$. Then $\delta_k=\delta_k^{(1)}+\delta_k^{(2)}$, $k=1,2,3,\ldots$, is a solution of (3.16) corresponding to boundary value

$$y_1(t)=[y_1^{(1)}(t)I_1(t)+y_1^{(2)}(t)I_2(t)]/[I_1(t)+I_2(t)] \tag{3.17}$$

where $I_j(t)=\exp[\int y_1^{(i)}\,\mathrm{d}t]$.

Proof. The δ_k are obviously solutions of (3.16). Also from (3.12) with $k=1$,

$$y_1=\frac{\omega_1}{\omega_0}=\frac{\dot{\delta}_1}{\delta_1}=\frac{\dot{\delta}_1^{(1)}+\dot{\delta}_1^{(2)}}{\delta_1^{(1)}+\delta_1^{(2)}} \tag{3.18}$$

since $\omega_0(t)=1$. The result then follows from the fact that

$$\delta_1^{(j)}=\exp\left[\int y_1^{(j)}\,\mathrm{d}t\right]=I_j(t). \tag{3.19}$$

Starting from any given $\delta_1(t)$, (3.16), (3.15) and (3.12) in that order lead to a solution of the KVM lattice equations. A simple non-trivial case is when we choose $\delta_1(t)=D^n\exp(-t^2)$, $n=0,1,2,\ldots$, with $D=\mathrm{d}/\mathrm{d}t$. Then

$$\omega_k(t)=(-1)^k\left[\frac{(-1)^{k+n}\exp(t^2)D^{k+n}\exp(-t^2)}{(-1)^n\exp(t^2)D^n\exp(-t^2)}\right]$$

$$=(-1)^k\frac{H_{k+n}(t)}{H_n(t)} \qquad k,n=0,1,\ldots \tag{3.20}$$

where we have used Roderigue's formula for Hermite polynomials $H_j(t)$. For each value of $n=0,1,2,\ldots$, we then obtain rational solutions of the KVM equations.

In their original paper Kac and Van Moerbeke (1975) showed that solutions of their lattice equations give solutions of the Toda lattice equations. The result may be stated as follows.

Theorem 3.3. When $y_k(t)$, $k=1,2,3,\ldots$, satisfy the KVM lattice equations (3.2) and (3.3), then

$$M_k(t)=y_{2k}(t)y_{2k-1}(t) \qquad N_k(t)=y_{2k+1}(t)+y_{2k}(t) \qquad k=1,2,\ldots \tag{3.21}$$

satisfy the Toda lattice equations

$$\dot{N}_k(t)=M_{k+1}(t)-M_k(t) \tag{3.22a}$$

$$\dot{M}_{k+1}(t)M_{k+1}^{-1}(t)=N_{k+1}(t)-N_k(t) \tag{3.22b}$$

for $k=1,2,\ldots$, and also the boundary condition

$$N_1(t)=\dot{M}_1(t)M_1^{-1}(t)+\int M_1\,\mathrm{d}t. \tag{3.23}$$

The proof is by straightforward substitution. This result may be used for example to generate rational solutions of the Toda lattice from the corresponding solutions of the KVM lattice obtained from (3.20).

4. Discrete nonlinear Schrödinger equation

The half-infinite DNLS equation is

$$\mathrm{i}\dot{Q}_k(t)=Q_{k+1}(t)+Q_{k-1}(t)-2Q_k(t)-|Q_k(t)|^2[Q_{k+1}(t)+Q_{k-1}(t)]$$
$$k=0,1,2,\ldots. \tag{4.1}$$

In the AKNS formalism, (4.1) is given by the consistency of the evolution equations:

$$\dot{\nu}_{1k}=\mathrm{i}[1-\lambda^2+Q^*_{k-1}Q_k]\nu_{1k}+\mathrm{i}[Q_{k-1}/\lambda-Q_k\lambda]\nu_{2k} \tag{4.2a}$$

$$\dot{\nu}_{2k}=\mathrm{i}[Q^*_k/\lambda-Q^*_{k-1}\lambda]\nu_{1k}+\mathrm{i}(1/\lambda^2-Q_{k-1}Q^*_k-1]\nu_{2k} \qquad k=0,1,2,\ldots \tag{4.2b}$$

with the eigenvalue equation:

$$\nu_{1k+1}=\lambda\nu_{1k}+Q_k\nu_{2k} \tag{4.3a}$$

$$\nu_{2k+1}=Q^*_k\nu_{1k}+\nu_{2k}/\lambda \qquad k=0,1,2,3,\ldots. \tag{4.3b}$$

Setting $Z_k(t)=\nu_{1k}(t)/\nu_{2k}(t)$ we can generate from (4.3a, b) the scalar continued-fraction expansion:

$$Z_0=-\frac{1}{(Q^*_0\lambda)}-\frac{Q_0/Q^*_0-1/Q^{*2}_0}{\lambda/Q^*_0+1/(Q^*_1\lambda)+}\,\frac{Q_1/Q^*_1-1/Q^{*2}_1}{\lambda/Q^*_1+1/(Q^*_2\lambda)+\ldots} \tag{4.4}$$

which from (4.2a, b) is a solution of the Riccati equation:

$$\dot{Z}_0=\mathrm{i}[Q_{-1}/\lambda-Q_0\lambda]+\mathrm{i}[2-\lambda^2-1/\lambda^2+Q^*_{-1}Q_0+Q_{-1}Q^*_0]Z_0$$
$$+\mathrm{i}[Q^*_{-1}\lambda-Q^*_0/\lambda]Z_0^2. \tag{4.5}$$

Just as for the KVM lattice discussed in the previous section, this equation can be transformed into a single linear differential equation when a simple boundary condition is satisfied.

Theorem 4.1. If the boundary condition $Q_{-1}(t)=\exp(2\mathrm{i}t)$ holds, then $W(t)\equiv[Z_0(t)-\lambda\exp(2\mathrm{i}t)]^{-1}$ satisfies the linear equation

$$\dot{W}=\mathrm{i}Q^*_0/\lambda-\mathrm{i}\exp(-2\mathrm{i}t)\lambda-\mathrm{i}[2+\lambda^2+1/\lambda^2+Q_0\exp(-2\mathrm{i}t)-Q^*_0\exp(2\mathrm{i}t)]W. \tag{4.6}$$

The proof follows from substituting for Z_0 in terms of W in (4.5) and then using the given boundary condition.

Using an equivalence relation for the continued fraction (4.4) for Z_0 and substituting in the definition of W we find

$$W=\frac{1}{\lambda}\left[\frac{\gamma F_1}{1+G_1\gamma+}\,\frac{\gamma F_2}{1+G_2\gamma+}\,\frac{\gamma F_3}{1+G_3\gamma+\ldots}\right] \qquad \gamma=\lambda^2 \tag{4.7}$$

with

$$F_1\equiv Q^*_0 \qquad G_1\equiv\exp(2\mathrm{i}t)Q^*_0$$
$$F_n\equiv Q^*_{n-1}(Q_{n-2}-1/Q^*_{n-2}) \qquad G_n\equiv Q^*_{n-1}/Q^*_{n-2} \qquad n=2,3,4,\ldots. \tag{4.8}$$

The expression in brackets in (4.7) has been called a T-fraction by Jones and Thron (1980). It corresponds to a pair of formal power series expansions in γ and in γ^{-1}, i.e. given two such series

$$L=\sum_{k=1}^{\infty}\mu_k\gamma^k \qquad \tilde{L}=\sum_{k=-\infty}^{0}\tilde{\mu}_k\gamma^k \tag{4.9}$$

there exists a T-fraction:

$$\frac{F_1\gamma}{1+G_1\gamma+}\frac{F_2\gamma}{1+G_2\gamma+}\frac{F_3\gamma}{1+G_3\gamma+}\dots$$

such that, when it is expanded in powers of γ it gives L, and when it is expanded in powers of γ^{-1} it gives $\tilde{L}$. The elements of the continued fraction are given by

$$F_n=-\frac{H_{n-2}^{(-n+3)}H_n^{(-n+2)}}{H_{n-1}^{(-n+2)}H_{n-1}^{(-n+3)}}\qquad G_n=-\frac{H_{n-1}^{(-n+1)}H_n^{(-n+2)}}{H_n^{(-n+1)}H_{n-1}^{(-n+3)}}\tag{4.10}$$

where $H_m^{(n)}$ are the Hankel determinants defined in (3.14) but with ω_n replaced by δ_n defined by

$$\begin{aligned}\delta_n&=-\mu_n\qquad n=1,2,\dots\\&=\tilde{\mu}_n\qquad n=0,-1,-2,\dots.\end{aligned}\tag{4.11}$$

We are now ready to prove the main result of this section.

Theorem 4.2. The DNLS equation (4.1) has the following solution for given $Q_0(t)$ when the boundary condition $Q_{-1}(t)=\exp(2\mathrm{i}t)$ holds:

$$Q_n(t)=(-1)^{n+1}\exp(2\mathrm{i}t)\left[\frac{H_{n+1}^{(-n+1)}}{H_{n+1}^{(-n)}}\right]^*\qquad n=1,2,\dots\tag{4.12}$$

where the Hankel determinants are those defined above corresponding to the δ_n and μ_k, $\tilde{\mu}_k$ are solutions of the linear equations

$$\begin{aligned}&\mu_{k+1}=[2+Q_0\exp(-2\mathrm{i}t)-Q_0^*\exp(2\mathrm{i}t)]\mu_k-\mathrm{i}\dot{\mu}_k+\mu_{k-1}\qquad k=2,3,\dots\\&\mu_1=Q_0^*\qquad \mu_2=2Q_0^*+|Q_0|^2\exp(-2\mathrm{i}t)-Q_0^{*2}\exp(2\mathrm{i}t)-\exp(-2\mathrm{i}t)-\mathrm{i}\dot{Q}_0^*\end{aligned}\tag{4.13}$$

and

$$\begin{aligned}&\tilde{\mu}_{k-1}=\tilde{\mu}_{k+1}-[2+Q_0\exp(-2\mathrm{i}t)-Q_0^*\exp(2\mathrm{i}t)]\tilde{\mu}_k+\mathrm{i}\dot{\tilde{\mu}}_k\qquad k=-2,-3,\dots\\&\tilde{\mu}_0=\exp(-2\mathrm{i}t)\qquad \tilde{\mu}_{-1}=-Q_0\exp(-4\mathrm{i}t)\\&\tilde{\mu}_{-2}=Q_0^2\exp(-4\mathrm{i}t)-\mathrm{i}\dot{Q}_0\exp(-4\mathrm{i}t)-2Q_0\exp(-4\mathrm{i}t)-|Q_0|^2\exp(-2\mathrm{i}t)+\exp(2\mathrm{i}t).\end{aligned}\tag{4.14}$$

Proof. It is straightforward to prove by equating powers of λ that $W=-L/\lambda$ is a solution of (4.6) when (4.13) hold. Similarly $W=-\tilde{L}/\lambda$ is a solution of (4.6) when (4.14) hold. From (4.8)

$$Q_n^*=\exp(-2\mathrm{i}t)\prod_{m=1}^{n+1}G_m\qquad n=1,2,\dots\tag{4.15}$$

and (4.12) follows on using (4.10).

The linear equations for the μ_k and $\tilde{\mu}_k$ may be simplified as was done for the ω_k in the KVM case.

Corollary 4.2.

$$\mu_k=-\exp(-2\mathrm{i}t)\sigma_{k+1}/\sigma_1\qquad k=1,2,\dots\tag{4.16}$$

$$\tilde{\mu}_k=\exp(-2\mathrm{i}t)\tilde{\sigma}_{k-1}/\tilde{\sigma}_{-1}\qquad k=0,-1,-2,\dots\tag{4.17}$$

are solutions of (4.13) and (4.14), respectively, when

$$\mathrm{i}\dot{\sigma}_{k+1} = \sigma_k - \sigma_{k+2} \qquad k = 1, 2, 3, \ldots \tag{4.18a}$$

$$-\mathrm{i}\dot{\sigma}_1 = [Q_0 \exp(-2\mathrm{i}t) - Q_0^* \exp(2\mathrm{i}t)]\sigma_1 \tag{4.18b}$$

$$\sigma_2 = -Q_0^* \exp(+2\mathrm{i}t)\sigma_1 \tag{4.18c}$$

and

$$\mathrm{i}\dot{\tilde{\sigma}}_{k-1} = \tilde{\sigma}_{k-2} - \tilde{\sigma}_k \qquad k = -1, -2, -3, \ldots \tag{4.19a}$$

$$-\mathrm{i}\dot{\tilde{\sigma}}_{-1} = [Q_0 \exp(-2\mathrm{i}t) - Q_0^* \exp(2\mathrm{i}t)]\tilde{\sigma}_{-1} \tag{4.19b}$$

$$\tilde{\sigma}_{-2} = -Q_0 \exp(-2\mathrm{i}t)\tilde{\sigma}_{-1}. \tag{4.19c}$$

Equations (4.18) and (4.19) are the final form of the linearisation of the DNLS with boundary condition $Q_{-1} = \exp(2\mathrm{i}t)$. Thus, given Q_0, we may use these equations to compute the σ_k and $\tilde{\sigma}_k$ and hence through (4.16) and (4.17) the μ_k and $\tilde{\mu}_k$. The Q_n are finally obtained from (4.12).

A simple non-trivial solution of (4.18), (4.19) in terms of Bessel functions of the first kind is obtained by taking

$$Q_0 = -[1 + \tfrac{1}{2}J_0(-2\mathrm{i}t)] \exp(2\mathrm{i}t). \tag{4.20}$$

Since $J_0(-2\mathrm{i}t)$ is real for real t, we may take

$$\sigma_1(t) = 1 = \tilde{\sigma}_{-1}(t) \tag{4.21}$$

this choice being consistent with (4.18b) and (4.19b). Substituting in (4.18c) and (4.19c)

$$\sigma_2(t) = 1 + \tfrac{1}{2}J_0(-2\mathrm{i}t) = \tilde{\sigma}_{-2}(t). \tag{4.22}$$

Using the recurrence relation:

$$J_{k-1}(x) - J_{k+1}(x) = 2\frac{\mathrm{d}J_k(x)}{\mathrm{d}x} \tag{4.23}$$

in the equations (4.18a) and (4.19a), it is easy to show that

$$\sigma_k(t) = 1 + \sum_{n=1}^{[k/2]} A_{k,n} J_{k-2n}(-2\mathrm{i}t) \qquad k = 2, 3, \ldots \tag{4.24}$$

and

$$\tilde{\sigma}_{-k}(t) = 1 + (-1)^k \sum_{n=1}^{[k/2]} A_{k,n} J_{k-2n}(-2\mathrm{i}t) \qquad k = 2, 3, \ldots \tag{4.25}$$

where

$$\begin{aligned} A_{k,n} &= (-1)^{n+1} \qquad 2n \neq k \\ A_{k,k/2} &= \tfrac{1}{2}(-1)^{k/2+1} \qquad k \text{ even.} \end{aligned} \tag{4.26}$$

The corresponding solutions for $Q_n(t)$ may be obtained in the manner outlined above in the form of determinants whose elements are sums of Bessel functions.

5. Discrete modified KdV equation

Our discussion of the DMKdV

$$\dot{Q}_k(t)=[1-Q_k^2(t)][Q_{k+1}(t)-Q_{k-1}(t)] \tag{5.1}$$

follows that of the previous section. Ablowitz and Ladik (1976) have shown that this equation is given by the consistency of the evolution equations:

$$\dot{\nu}_{1k}=[\lambda^2-Q_{k-1}Q_k]\nu_{1k}+[Q_k\lambda+Q_{k-1}/\lambda]\nu_{2k} \tag{5.2a}$$

$$\dot{\nu}_{2k}=[Q_{k-1}\lambda+Q_k/\lambda]\nu_{1k}+[1/\lambda^2-Q_{k-1}Q_k]\nu_{2k} \tag{5.2b}$$

with the eigenvalue equations:

$$\nu_{1k+1}=\lambda\nu_{1k}+Q_k\nu_{2k} \tag{5.3a}$$

$$\nu_{2k+1}=Q_k\nu_{1k}+\nu_{2k}/\lambda. \tag{5.3b}$$

Setting $Z_k(t)\equiv\nu_{1k}(t)/\nu_{2k}(t)$, we can, as previously, construct the continued fraction:

$$Z_0(t)=-\frac{1}{\lambda Q_0}-\frac{(1-1/Q_0^2)}{\lambda/Q_0+1/(\lambda Q_1)+}\frac{(1-1/Q_1^2)}{\lambda/Q_1+1/(\lambda Q_2)+}\frac{(1-1/Q_2^2)}{\lambda/Q_2+1/(\lambda Q_3)+\ldots}. \tag{5.4}$$

From (5.2a, b), Z_0 satisfies the Riccati equation:

$$\dot{Z}_0=\lambda Q_0+Q_{-1}/\lambda+(\lambda^2-1/\lambda^2)Z_0-(Q_{-1}\lambda+Q_0/\lambda)Z_0^2. \tag{5.5}$$

Again this can be transformed into a single linear differential equation when a simple boundary condition is satisfied.

Theorem 5.1. Consider the DMKdV lattice with boundary condition $Q_{-1}(t)=1$. Then $W(t)\equiv[Z_0(t)-\lambda]^{-1}$ satisfies the linear equation:

$$\dot{W}=\lambda+Q_0/\lambda+[\lambda^2+1/\lambda^2+2Q_0]W. \tag{5.6}$$

The proof is by substituting for Z_0 in terms of W in (5.5).

As previously, W can be expressed in terms of a T-fraction:

$$W=-1/\lambda\left[\frac{F_0\gamma}{1+G_0\gamma+}\frac{F_1\gamma}{1+G_1\gamma+}\frac{F_2\gamma}{1+G_2\gamma+\ldots}\right] \qquad \gamma=\lambda^2 \tag{5.7}$$

where

$$F_1=G_1=Q_0 \tag{5.8a}$$

$$F_n=Q_{n-2}Q_{n-1}(1-1/Q_{n-1}^2) \qquad G_n=Q_{n-1}/Q_{n-2}. \tag{5.8b}$$

From (5.7), $-W$ has formal power series expansions $\Sigma_{k=1}^{\infty}\mu_k\lambda^{2k-1}$ and $\Sigma_{k=-\infty}^{0}\tilde{\mu}_k\lambda^{2k-1}$ and they are related to the F_n and G_n through (3.14), (4.10) and (4.11). Thus we can prove a result corresponding to theorem 4.2 of the previous section.

Theorem 5.2. The discrete modified KdV equation (5.1), with the boundary condition $Q_{-1}(t)\equiv 1$ has the following solution for given $Q_0(t)$:

$$Q_n(t)=(-1)^{n+1}\frac{H_{n+1}^{(-n+1)}}{H_{n+1}^{(-n)}} \qquad n=1,2,\ldots \tag{5.9}$$

where the $H_k^{(j)}$ are the Hankel determinants given by (4.10) and (4.11), with μ_k and $\tilde{\mu}_k$ solutions of the linear equations:

$$\begin{aligned} &\mu_{k+1}=\dot{\mu}_k-2Q_0\mu_k-\mu_{k-1} \qquad k=2,3,4,\ldots \\ &\mu_1=Q_0 \qquad \mu_2=\dot{Q}_0-2Q_0^2+1 \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} &\tilde{\mu}_{k-1}=\dot{\tilde{\mu}}_k-2Q_0\tilde{\mu}_k-\tilde{\mu}_{k+1} \qquad k=-2,-3,-4,\ldots \\ &\tilde{\mu}_0=1 \qquad \tilde{\mu}_{-1}=-Q_0 \qquad \tilde{\mu}_{-2}=-\dot{Q}_0+2Q_0^2-1. \end{aligned} \tag{5.11}$$

The proof of this theorem follows that of theorem 4.2 and (5.10) and (5.11) follow from substituting the above power series for W in (5.6) and equating powers of λ. Again the above equations for μ_k and $\tilde{\mu}_k$ may be simplified.

Corollary 5.2.

$$\mu_k=-\sigma_{k+1}/\sigma_1 \qquad k=1,2,\ldots \tag{5.12a}$$

$$\tilde{\mu}_k=\tilde{\sigma}_{k-1}/\tilde{\sigma}_{-1} \qquad k=0,-1,-2,\ldots \tag{5.12b}$$

are solutions of (5.10) and (5.11) when

$$\dot{\sigma}_{k+1}=\sigma_{k+2}+\sigma_k \qquad k=1,2,\ldots \tag{5.13a}$$

$$\dot{\sigma}_1=-2Q_0\sigma_1 \tag{5.13b}$$

$$\sigma_2=-Q_0\sigma_1 \tag{5.13c}$$

and

$$\dot{\tilde{\sigma}}_{k-1}=\tilde{\sigma}_{k-2}+\tilde{\sigma}_k \tag{5.14a}$$

$$\dot{\tilde{\sigma}}_{-1}=-2Q_0\tilde{\sigma}_{-1} \tag{5.14b}$$

$$\tilde{\sigma}_{-2}=-Q_0\tilde{\sigma}_{-1}. \tag{5.14c}$$

This formulation may, as in the corresponding ones for the KVM and DNLS lattices, be used to generate a non-trivial solution of the DMKdV equations in terms of standard functions. It corresponds to taking

$$Q_0(t)=-I_1(2t)/I_0(2t) \qquad Q_{-1}(t)=1 \tag{5.15}$$

where $I_k(x)$ are modified Bessel functions. Using the recurrence relation,

$$I_{k-1}(x)+I_{k+1}(x)=2\frac{\mathrm{d}I_k(x)}{\mathrm{d}x} \tag{5.16}$$

the system of equations (5.13) and (5.14) may be solved to give

$$\sigma_k(t)=I_{k-1}(2t)=\tilde{\sigma}_{-k}(t) \qquad k=1,2,\ldots. \tag{5.17}$$

The solutions for Q_n are then obtained in the usual way as quotients of determinants whose elements are those modified Bessel functions.

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